A planar start-up demonstration procedure

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$M=3,\ N=6\ , k_s=10, k_{cs}=3, k_f=8, k_{cf}=2$

s	f	f	s	s	s
f	f	f	s	s	s
f	f	f	s	s	s

Example: acceptance rectangle M=2, kcs=4

S	S	S	S		
S	S	S	S		

Example: M=2, k_s=7, k_{cs}=2, k_f=3

s	f	f	s	s
f	s	s	s	s

Rejection: M=2, kcf=2

f	f		
f	f		

Acceptance/ Rejection rule

- Acceptance: either total of *ks* successes
- or both units have successes along the same consecutive *kcs* tests (so that we have a rectangular grid of Mx*k*cs successes).

- Rejection: either total of *kf* failures
- or both units fail at the same consecutive *k*cf tests (rectangular grid consisting of Mx*k*cf failures).

M=2 units

- Columns representing last stage results:

$$V = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$$f(i,j,n) = P\left\{T_{n,s} = i, R_{M,n,s} < Mk_{cs}, R_{M,n,f} < Mk_{cf}, \underline{X}_n = \underline{V}_j\right\}$$

Inter-connections between f values

$$For\ 1 < j < 2^M, n > 1:$$

$$f(i,j,n) = p^{\left\|\underline{V}_j\right\|} q^{M-\left\|\underline{V}_j\right\|} \cdot \sum_{j'=1}^{2^M} f\left(i - \left\|\underline{V}_j\right\|, j', n-1\right)$$

Number of parallel tests= N

$$P\{N > n\} = \sum_{i=\max(Mn-k_f+1,0)}^{\min(k_s-1,Mn)} \sum_{j=1}^{2^M} f(i,j,n)$$

$$P\{N = n\} = P\{N > n-1\} - P\{N > n\}$$

Probability of Acceptance (P_a)

(1) Reaching the total number of k_s successes before k_f failures and rectangle of Mk_{cf} failures

(2) Existence of a rectangle of Mk_{cs} successes before k_f failures and of Mk_{cf} rectangle of failures

Constrained optimization problem

Specify: α, β, p_U, p_L

Minimize: expected number of required tests

Find values of: *ks, kcs, kf, kcf*

Constraints:

$$P\{acceptance \mid p = p_U\} > 1 - \beta$$

$$P\{acceptance \mid p = p_L\} < \alpha$$

CSTF, pU=0.9, pL=0.6, α=β=5%

M	k_{cs}	k_f	$P\{a/p_U\}$	$P\{a/p_L\}$	$E\{N\}$	Sd{N}
1	10	7	0.9503	0.0416	17.75	8.94
2	5	10	0.9635	0.0469	9.47	5.23
3	4	13	0.9549	0.0186	8.95	5.42
4	3	13	0.9522	0.0159	7.04	4.40
5	2	10	0.9690	0.0297	4.42	2.79

TSCSTFCF, pU=0.9, pL=0.6, α=β=5%

M	k_s	k_{cs}	k_f	k_{cf}	P{a/p_U}	P{a/p_L}	E{N}	Sd{N}
1	22	10	7	3	0.9741	0.0497	16.25	6.34
2	27	5	9	2	0.9531	0.0459	8.99	4.18
3	33	4	13	∞	0.9644	0.0490	8.14	3.79
4	33	4	13	∞	0.9814	0.0392	6.25	2.89
5	20	2	7	∞	0.9821	0.0440	3.54	1.33

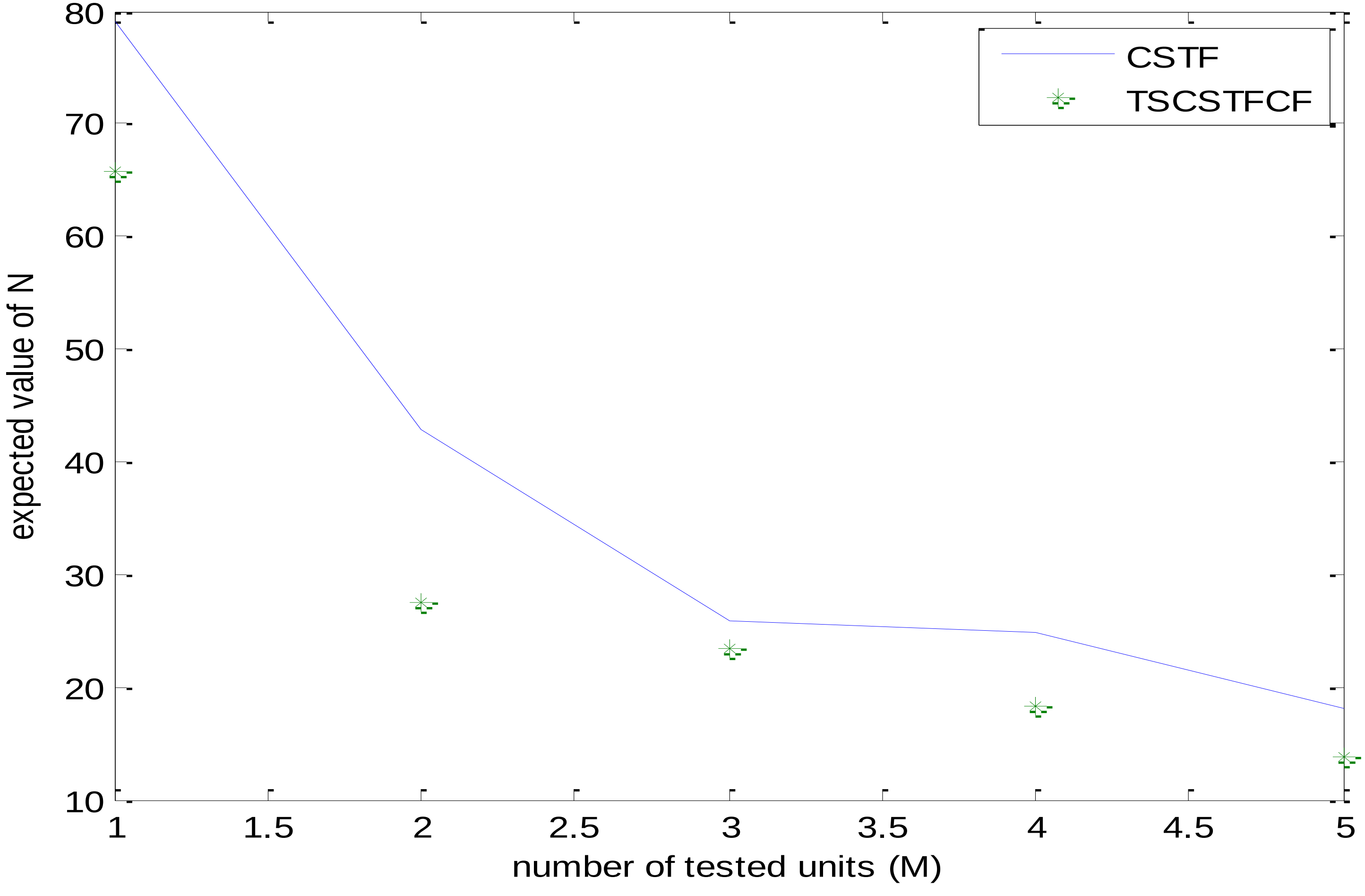
CSTF, *pU*=0.85, *pL*=0.65, α=β=5%

M	k_{cs}	k_f	$P\{a/p_U\}$	$P\{a/p_L\}$	$E\{N\}$	Sd{N}
1	16	39	0.9506	0.0388	79.01	58.41
2	8	45	0.9537	0.0368	42.85	32.42
3	5	41	0.9528	0.0433	25.80	19.44
4	4	51	0.9512	0.0301	24.81	19.20
5	3	51	0.9655	0.0399	18.13	14.46

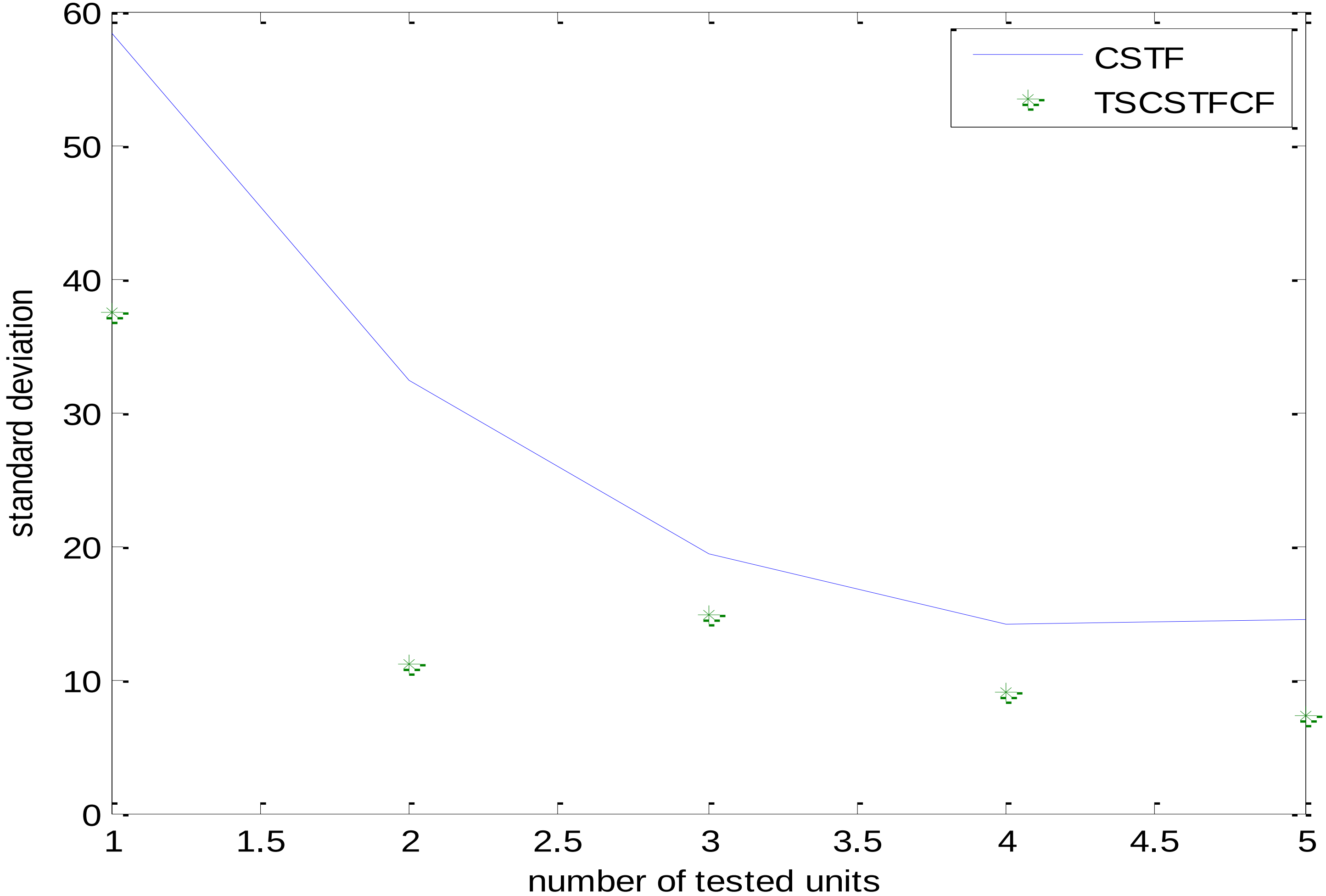
TSCSTFCF, pU=0.85, pL=0.65, α=β=5%

M	k_s	k_{cs}	k_f	k_{cf}	P{a/p_U}	P{a/p_L}	E{N}	Sd{N}
1	98	16	39	4	0.9716	0.0499	65.82	37.52
2	60	8	22	∞	0.9747	0.0478	27.54	11.11
3	120	5	31	∞	0.9549	0.0329	23.48	14.89
4	90	4	34	∞	0.9968	0.0432	18.21	9.04
5	90	3	33	∞	0.9975	0.0392	13.84	7.30

pU=0.85, pL=0.65, α=β=5%



pU=0.85, pL=0.65, α=β=5%



Conclusions

Extension of TSCSTFCF to two dimensions

Testing of several units in parallel.

Reasonable computation times are involved.

Constrained optimization problem: min. the expected number of tests subject to constraints on the confidence level of the probability of acceptance of the unit.

Running several units in parallel

reduces the expected number of tests compared to those of the single unit procedures

Practically, it seems worthwhile to test two units in parallel instead of a single one.

The more general TSCSTFCF should be used instead of CSTF in all cases.

- Reference: “Start-up demonstration tests involving a two-dimmensional TSCSTFCF procedure”, International Journal of Reliability, Quality and Safety Engineering, vol. 22, no. 1, 1550003, Feb. 2015.

A PLANAR START-UP DEMONSTRATION PROCEDURE

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Introduction
An extension of the previously introduced TSCSTFCF start-up demonstration procedure for the two-dimensional case is presented (see [2-7] for the one-dimensional case). It involves the testing of several units in parallel. A constrained optimization problem is set up for reducing the expected time of the test procedure subject to constraints on the confidence level on the probability of acceptance of the unit. Running several units in parallel shows the foreseen advantage in reducing the time of the duration of the tests for the present demonstration procedure compared to those of the single unit procedures.

Start-up demonstration procedures have been considered within various articles (a survey is given in [7]). In nearly all of them, a single unit is tested and it is of interest to know how many tests are needed for either accepting or rejecting the equipment. It is reasonable that running in parallel two or more units will shorten the expected time for the test procedure which seems to be a rather important design factor.

Reasonable computation times are involved in the analysis and the design procedures. Further on, it is intended to carry out a global optimization scheme which surely will yield even better results. In planning a set of start-up demonstration tests, the possibility of using several units for testing instead of a single one should be taken into account mainly for cutting short the time duration of the procedure. This may be practical in those cases when the testing time is a critical issue

The equations
Let M be the number of tested units. For any $M \geq m \geq 1$, let

$$u = 2^{M-m}$$

Define a matrix in the following way: for any $M \geq m \geq 1$, $2^M \geq j \geq 1$, and any $2^M \geq r \geq 0$,

$$V(m,j) = \begin{cases} 0 & (2r+1)u \geq j > 2ru \\ 1 & (2r+2)u \geq j > (2r+1)u \end{cases} \tag{1}$$

For instance, for M=2 units,

$$V = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

For any column j, we denote the relevant column vector by $\underline{V}_j$ and by $\|\cdot\|$ its L_1 norm.

Following Gera [2-6] ,a probability function is defined as follows:

$$f(i,j,n) = P\{T_{n,s} = i, R_{M,n,s} < Mk_{cs}, R_{M,n,f} < Mk_{cf}, \underline{X}_n = \underline{V}_j\} \tag{2}$$

where $T_{n,s}$ denotes the number of successes till the n'th set of parallel tests;
$R_{M,n,s}$, $R_{M,n,f}$ denote the maximal rectangles of successes (failures) up to the n'th set of parallel tests;
k_{cs}, k_{cf} represent the common number of consecutive successes within $R_{M,n,s}$, $R_{M,,n,f}$ yielding the acceptance (rejection) of the equipment;
X_n is the outcome vector of the n-th start-up tests' results for all units (=1 for success, =0 for failure)
This function may be evaluated through the following set of recursive equations (u[.],δ[.] are the unit step and impulse functions, $\| \ \|$ is the L_1 norm of vector):

$i \geq 1, n > 1$:

$$f(i,1,n) = \sum_{a=1}^{\min(k_{cf},n)-1} q^{Ma} \sum_{j'>1} f(i,j',n-a) \cdot u[M(n-a)-i] \tag{3}$$

$i \geq 1, n > 1$:

$$f\left(i, 2^M, n\right) = \sum_{b=1}^{\min\left(k_{cs}, n\right)-1} p^{Mb} \sum_{j' < 2^M} f\left(i - Mb, j', n - b\right) + p^{Mn} \cdot u\left[k_{cs} - 1 - n\right] \cdot \delta[i - Mn] \tag{4}$$

$$For\ 1 < j < 2^M, n > 1: \tag{5}$$

$$f(i, j, n) = p^{\left\|\underline{V}_j\right\|} q^{M - \left\|\underline{V}_j\right\|} \cdot \sum_{j'=1}^{2^M} f\left(i - \left\|\underline{V}_j\right\|, j', n - 1\right)$$

Appropriate boundary conditions are added.

The distribution function for the number of sets of parallel tests *N* will be:

$$P\{N > n\} = \sum_{i=\max\left(Mn - k_f + 1, 0\right)}^{\min\left(k_s - 1, Mn\right)} \sum_{j=1}^{2^M} f(i, j, n) \tag{6}$$

and the probability mass distribution function is given by

$$P\{N = n\} = P\{N > n - 1\} - P\{N > n\} \tag{7}$$

The probability of acceptance of the equipment is also of interest. Let k_s be the total number of successes required for the acceptance of the equipment. The probability of acceptance due to reaching the total number of k_s successes is given by

$$P_{a,1,n} = P\left\{T_{n,s} \geq k_s, T_{n-1,s} < k_s, T_{n,f} < k_f, R_{M,n,s} < Mk_{cs}, R_{M,n,f} < Mk_{cf}\right\} \tag{8}$$

Otherwise, the equipment may be accepted if there exists a rectangle of successes with the probability:

$$P_{a,2,n} = P\left\{any\ \ T_{n,s}, R_{M,n-1,s} < Mk_{cs}, R_{M,n,s} = Mk_{cs}, T_{n,f} < k_f, R_{M,n,f} < k_{cf}\right\} \tag{9}$$

It is observed that necessarily

$$T_{n-1,s} = T_{n,s} - \left\|\underline{X}_n\right\| < k_s \tag{10}$$

Also, it is required that

$$T_{n,f} = Mn - T_{n,s} < k_f \tag{11}$$

Therefore,

$$\max\left(k_s - 1, Mn - k_f\right) < T_{n,s} < k_s + \left\|\underline{X}_n\right\| \tag{12}$$

Let $P_{a,1,j,n} = P\left\{T_{n,s} \geq k_s, T_{n-1,s} < k_s, T_{n,f} < k_f, R_{M,n,s} < Mk_{cs}, R_{M,n,f} < Mk_{cf}, \underline{X}_n = \underline{V}_j\right\}$ (13)

so that, for a certain termination vector $\underline{V}_j$,

$$P_{a,1,j,n} = \sum_{i=\max\left(k_s-1, Mn-k_f\right)+1}^{k_s+\left\|\underline{X}_n\right\|-1} f(i, j, n) \tag{14}$$

The total probability of acceptance in this case will be given by:

$$P_{a,1} = \sum_{n=1}^{\infty} \sum_{j=2}^{2^M} P_{a,1,j,n} \tag{15}$$

The second way of achieving the acceptance of the equipment is as follows:

Let

$$P_{a,2,i,n} = P\left\{T_{n,s} = i, R_{M,n-1,s} < Mk_{cs}, R_{M,n,s} = Mk_{cs}, T_{n,f} < k_f, R_{M,n,f} < Mk_{cf}\right\} \tag{16}$$

then

$$P_{a,2,i,n} = p^{Mk_{cs}} \cdot P\left\{T_{n-k_{cs},s} = i - Mk_{cs}, T_{n-1,s} < k_s, R_{M,n-k_{cs},s} < Mk_{cs}, T_{n-k_{cs},f} < k_f, R_{M,n-k_{cs},f} < Mk_{cf}, \underline{X}_{n-k_{cs}} \neq \underline{V}_{2^M}\right\} \tag{17}$$

so that

$$P_{a,2} = \sum_{n=k_{cs}}^{\infty} \sum_{i=\max\left(0, Mn-k_f+1, Mk_{cs}\right)}^{n} P_{a,2,i,n}$$

The upper bounds in these sums may be replaced by:

$$P_{a,2} = \sum_{n=k_{cs}}^{n_2} \sum_{i=\max\left(0, Mn-k_f+1, Mk_{cs}\right)}^{\min(n, k_s+M-1)} P_{a,2,i,n} \tag{18}$$

where

$$n_2 = \begin{cases} \dfrac{k_s + k_f + M - 1}{M} - 1 = r & natural \quad r \\ \left\lceil \dfrac{k_s + k_f + M - 1}{M} \right\rceil & otherwise \end{cases}$$

After that, the probability of acceptance is simply

$$P_a = P_{a,1} + P_{a,2} \tag{19}$$

Constrained optimization

A reasonable design goal for the testing procedure is the minimization of the length of the test procedure subject to constraints on the confidence level of accepting the tested units. Smith and Griffith [11,12] have suggested a procedure for minimizing the expected number of tests for a single procedure through a correct choice of the k_{cs}, k_f parameters for the CSTF model. The optimization is carried out with respect to some constraints. Gera [2-6] treated the same problem for the more general TSCSTFCF model.

According to the above references, the equipment should be accepted if the resultant value of the identical probability of success (p) of each test is higher than some specific value p_U and it is rejected if that value is lower than some initially set value p_L. Here we will generalize this concept for M units that are tested in parallel. Explicitly, it is required that

$$P\{acceptance \mid p = p_U\} > 1-\beta \tag{20}$$

$$P\{acceptance \mid p = p_L\} < \alpha \tag{21}$$

It is then required to find the values of k_s, k_{cs}, k_f, k_{cf} that will minimize the expected number of required tests subject to the above constraints (20),(21) on the confidence level.

Numerical results

Using the model equations, the optimization problem has been solved. Results presented are meanwhile only sub-optimal. They are based on a reasonable initial guess (referring to known values for the M=1 case) and some further cut and try iterations. These sub-optimal results were sufficient in the past for showing the superiority of using the TSCSTFCF procedure for a single tested unit and they seem to be sufficient for our purpose of showing the benefit of testing in parallel more than a single unit. It is intended to provide in the future an algorithm yielding the global optimum. In the following, two examples are provided which present the values of the expected number of tests (E{N}) together with the probabilities of acceptance due to using the higher value of probability (p_U) and the lower one (p_L). Sd{N} stands for the standard deviation of E{N}. We compare the results of optimization using CSTF which involves only the k_{cs}, k_f design parameters and those owing to the TSCSTFCF procedure which includes also the k_s, k_{cf} parameters (tables 1-2).

Example: p_U=0.9, p_L=0.6, α=β=5%, CSTF (k_{cs},k_f)

Table 1: CSTF, p_U=0.9, p_L=0.6, α=β=5%

M	k_{cs}	k_f	P{a/p_U}	P{a/p_L}	E{N}	Sd{N}
1	10	7	0.9503	0.0416	17.75	8.94
2	5	10	0.9635	0.0469	9.47	5.23
3	4	13	0.9549	0.0186	8.95	5.42
4	3	13	0.9522	0.0159	7.04	4.40
5	2	10	0.9690	0.0297	4.42	2.79

Table 2: TSCSTFCF, p_U=0.9, p_L=0.6, α=β=5%

M	k_s	k_{cs}	k_f	k_{cf}	P{a/p_U}	P{a/p_L}	E{N}	Sd{N}
1	22	10	7	3	0.9741	0.0497	16.25	6.34
2	27	5	9	2	0.9531	0.0459	8.99	4.18
3	33	4	13	∞	0.9644	0.0490	8.14	3.79
4	33	4	13	∞	0.9814	0.0392	6.25	2.89
5	20	2	7	∞	0.9821	0.0440	3.54	1.33

Evidently, the main benefit of reducing the time for the testing procedure comes from using two units instead of a single one. If reduction of this time duration is the most significant issue, then the testing of five units is even better. Using the TSCSTFCF procedure further reduces $E\{N\}$ compared to the simpler CSTF model. The same goes with the values of the associated standard deviation.

Conclusions

The extension of the previously introduced TSCSTFCF start-up demonstration procedure for the two-dimensional case has been presented. It involves the testing of several units in parallel. Numerical results for the new model stand in full correlation to those obtained for the simpler models. It is implicitly assumed that the probabilities of success of each test are known.

Reasonable computation times are involved in the analysis and the design procedures. A constrained optimization has been set up for reducing the value of the expected number of tests subject to constraints on the confidence level on the probability of acceptance of the unit. Running several units in parallel shows the foreseen advantage in reducing the value of the expected number of tests for the present demonstration procedure compared to those of the single unit procedures is evident. The standard deviation results are much lower. Further on, it is intended to carry out a global optimization scheme which surely will yield even better results. The compromise between the number of tested units and the expected time length of the procedure is illustrated. In planning a set of start-up demonstration tests, the possibility of using several units for testing instead of a single one should be taken into account mainly for cutting short the time duration of the procedure. This may be practical in those cases when the testing time is a critical issue. Generally, it is observed that it is worthwhile to use the more general TSCSTFCF compared to CSTF.

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